THE BELL SYSTEM TECHNICAL JOURNAL

DEVOTED TO THE SCIENTIFIC AND ENGINEERING
ASPECTS OF ELECTRICAL COMMUNICATION

Volume 55 | September 1976 | Number 7

Traffic Capacity of a Probability-Engineered Trunk Group

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(Manuscript received January 22, 1976)

The grade of service of a probability-engineered trunk group is defined to be the 20-day average blocking during the busy hour of the busy season. In this paper, an improved model for calculating grade of service is developed and used to increase the accuracy of the existing trunk-engineering procedures. Using the new model, new traffic-capacity tables and trunk-estimation algorithms have been designed for use in the Bell System.

I. INTRODUCTION

The grade of service for a probability-engineered trunk group is defined to be the average blocking observed in the time-consistent busy hour of the busy season.* The existing methods for predicting grade of service do not account for the effects of the finite length of the individual one-hour measurement intervals and thereby tend to underestimate trunk-group capacity.

In this paper, we develop an improved model for calculating average blocking that includes the two essential effects of the finite measurement interval. First, the current method for estimating the mean blocking for a single hour must be revised to remove an implicit assumption that the measurement interval is infinite. Second, the existing mathematical model for day-to-day variation of trunk-group

* Both the CCITT (International Telegraph and Telephone Consultative Committee) and the Bell System define grade of service as an unweighted average of busy-hour busy-season blocking values.

offered loads must be modified to account for statistical measurement error, which is also introduced by the finite measurement interval.

We develop a new model of day-to-day load variation in Appendix A and in Section II combine it with a new estimate of mean single-hour blocking to obtain our approximation for the average blocking. This new approximation is then compared with the existing approximation analytically and numerically; the accuracy of the new approach is established in the third section using data from a computer simulation. A summary is given in the last section.

II. AVERAGE BLOCKING

In this section, we develop an approximation for the average busy-season busy-hour blocking. Since the measured grade of service is determined from blocking measurements made over several hours, and since the busy-hour source loads vary from day to day, our analysis must account for the effects of such load variation. Accordingly, we first discuss day-to-day load variation and then describe our approximation for average blocking.

2.1 Day-to-day load variation

R. I. Wilkinson was the first author to study the impact of day-to-day variation in offered loads on trunk-engineering procedures.[1,2] He collected data from a number of trunk groups that indicated that the distribution of the observed loads could be approximated by a gamma distribution. The data also indicated that the variance, Var $(\hat{\alpha})$, of the observed load, $\hat{\alpha}$, was related to the mean $\bar{\alpha}$ by

$$\text{Var}\ (\hat{\alpha}) = 0.13\bar{\alpha}^{\phi}, \tag{1}$$

where ϕ is a parameter whose value depends on local conditions.

Wilkinson's studies showed that Var $(\hat{\alpha})$ tends to be relatively larger for overflow traffic than for first-routed (Poisson) traffic. Those results led to the specification of three values of ϕ (1.5, 1.7, and 1.84) to cover a reasonable range of engineering applications. The *level* of day-to-day variation is called low when $\phi = 1.5$ is appropriate, medium for $\phi = 1.7$, and high when $\phi = 1.84$.

In Appendix A, we show that the variance of the observed "single-hour" offered loads is a sum of two components:

$$\text{Var}\ (\hat{\alpha}) = \frac{2\bar{\alpha}z}{(t/h)} + \text{Var}\ (\alpha), \tag{2}$$

where α is the daily source load (a random variable), $\hat{\alpha}$ is the observed load, $\bar{\alpha} = E(\alpha)$ is the average busy-season load, z is the traffic peakedness, t is the observation interval (usually one hour), and h is the mean holding time. The first term is an approximation for the variance arising from the finite measurement interval* and Var (α) is the variance of the daily source load.

Combining eqs. (1) and (2), we obtain a model relating the source-load variation to the observed load variation; i.e.,

$$\text{Var}\ (\alpha) = 0.13\bar{\alpha}^{\phi} - \frac{2\bar{\alpha}z}{(t/h)}. \tag{3}$$

For certain combinations of the various parameters, $2\bar{\alpha}z/(t/h)$ can exceed $0.13\bar{\alpha}^{\phi}$, indicating that the observed load variation is entirely due to random measurement error resulting from the finite measurement interval. For our application, we will assume that

$$\text{Var}\ (\alpha) = \max \left\{0,\ 0.13\bar{\alpha}^{\phi} - \frac{2\bar{\alpha}z}{(t/h)}\right\}. \tag{4}$$

2.2 The approximation

Consider a service system with c servers having exponentially distributed service times with mean h and serving traffic under a blocked-calls-cleared service discipline. The system is observed during n disjoint measurement intervals $I_1, \cdots, I_n$, each of length t. During I_k, the interarrival times are independent and identically distributed (iid) with mean $1/\lambda_k$. The peakedness, z, of the traffic is assumed to be the same during all the intervals.† The system is in statistical equilibrium during each interval and the initial point of each interval is a stationary (random) point for the arrival process. The loads $\alpha_i = \lambda_i h$, $i = 1, \cdots, n$, are independent and identically distributed according to the distribution function $\Gamma(\alpha | \bar{\alpha}, v_d)$ with mean $\bar{\alpha}$ and variance $v_d = \text{Var}\ (\alpha)$ (the day-to-day source-load variance). [We assume that $\Gamma(\alpha | \bar{\alpha}, v_d)$ is a gamma distribution. Justification for the assumption is given below.]

We use $A_j(t)$ and $O_j(t)$ to denote, respectively, the number of arrivals (call attempts) and the number of blocked attempts (the overflows) during I_j; the (measured) observed blocking is $B_j = O_j(t)/A_j(t)$ provided $A_j(t) \neq 0$. If $A_j(t) = 0$, then $O_j(t) = 0$ and the ratio is not defined. However, if no arrivals occur during I_j, it seems appropriate to say that no blocking occurred; i.e., we define $B_j = 0$ whenever $A_j(t) = 0$. The sample average of the observed blocking (the observed grade of service) is

$$\bar{B}_n = \frac{1}{n} \sum_{j=1}^{n} B_j.$$

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* This is an extension of the formula given by Riordan in Ref. 3.

† Traffic studies have shown that the peakedness of the traffic offered to a final trunk group does not change significantly from day to day during the busy hour of the busy season.

Since $\alpha_1, \cdots, \alpha_n$ are iid, $B_1, \cdots, B_n$ are also iid. Consequently, the average blocking $\bar{B}$ is given by

$$\bar{B} = E(\bar{B}_n) = E(B_1). \tag{5}$$

The hourly measurement B_1 represents a random sample of observed blocking corresponding to a load population with mean α_1. The parameter α_1 is a random variable with mean $\bar{\alpha}$ and distributed according to $\Gamma(\alpha_1|\bar{\alpha}, v_d)$. Thus,

$$\bar{B} = \int_0^\infty E\left\{\frac{O_1(t)}{A_1(t)}\bigg|\alpha_1\right\} d\Gamma(\alpha_1|\bar{\alpha}, v_d). \tag{6}$$

Dropping the subscripts, the conditional mean in (6) is given by

$$E\left\{\frac{O(t)}{A(t)}\bigg|\alpha\right\} = \Pr\{A(t) > 0|\alpha\} E\left\{\frac{O(t)}{A(t)}\bigg|\alpha, A(t) > 0\right\},$$

where $\Pr(X)$ denotes the probability of the event X. Let $\bar{A} = E\{A(t)|\alpha\}$ and $\bar{O} = E\{O(t)|\alpha\}$. Also, for nonnegative m and n,

$$E\{[O(t)]^m[A(t)]^n|\alpha, A(t) > 0\} = \frac{E\{[O(t)]^m[A(t)]^n|\alpha\}}{\Pr\{A(t) > 0|\alpha\}}.$$

Using these relations, an approximation for the conditional expectation is obtained by expanding the function x/y in a two-dimensional Taylor Series about the point $(x_0, y_0) = (E\{O(t)|\alpha, A(t) > 0\}, E\{A(t)|\alpha, A(t) > 0\})$. Taking the appropriate conditional expectation and retaining only the terms up through second order, we have the approximation

$$\begin{aligned} E\left\{\frac{O(t)}{A(t)}\bigg|\alpha, A(t) > 0\right\} \approx \frac{\bar{O}}{\bar{A}} + \frac{\bar{O}}{\bar{A}}\left(\frac{\Pr\{A(t) > 0|\alpha\}}{\bar{A}}\right)^2 \\ \cdot\left[\frac{E\{[A(t)]^2|\alpha\}}{\Pr\{A(t) > 0|\alpha\}} - \left(\frac{\bar{A}}{\Pr\{A(t) > 0|\alpha\}}\right)^2\right] \\ - \left(\frac{\Pr\{A(t) > 0|\alpha\}}{\bar{A}}\right)^2 \\ \cdot\left(\frac{E\{A(t)O(t)|\alpha\}}{\Pr\{A(t) > 0|\alpha\}} - \frac{\bar{A}\bar{O}}{(\Pr\{A(t) > 0|\alpha\})^2}\right). \end{aligned} \tag{7}$$

Denoting the call congestion $\bar{O}/\bar{A}$ by $B(c, \alpha, z)$ and noting that $E\{A(t)|\alpha\} = \alpha t/h$, eq. (7) reduces to

$$\begin{aligned} E\left\{\frac{O(t)}{A(t)}\bigg|\alpha, A(t) > 0\right\} \approx B(c, \alpha, z) + \frac{\Pr\{A(t) > 0|\alpha\}}{(\alpha t/h)^2} \\ \cdot [B(c, \alpha, z)\operatorname{Var}\{A(t)|\alpha\} - \operatorname{Cov}\{A(t), O(t)|\alpha\}], \end{aligned}$$

where all of the statistical moments are functions of α, z, and c. Thus,

$$E\left\{\frac{O(t)}{A(t)}\middle|\alpha\right\} \approx \Pr\{A(t) > 0|\alpha\}B(c, \alpha, z) + \left(\frac{\Pr\{A(t) > 0|\alpha\}}{\alpha t/h}\right)^2 \cdot [B(c, \alpha, z)\,\mathrm{Var}\,\{A(t)|\alpha\} - \mathrm{Cov}\,\{A(t), O(t)|\alpha\}]. \quad (8)$$

Formulas for computing the moments are given in Ref. 4. An expression for $\Pr\{A(t) > 0|\alpha\}$ is given in Appendix B. Numerical experimentation has shown that $\Pr\{A(t) > 0|\alpha\} \approx 1$ for $\alpha t/h > 10$; i.e., that term can be ignored except for very small loads or short measurement intervals. Combining eqs. (6) and (8), we have

$$\bar{B} \approx \int_0^\infty \Pr\{A(t) > 0|\alpha\}B(c, \alpha, z)d\Gamma(\alpha|\bar{\alpha}, v_d) + \int_0^\infty R(c, \alpha, z)d\Gamma(\alpha|\bar{\alpha}, v_d), \quad (9)$$

where

$$R(c, \alpha, z) = \left(\frac{\Pr\{A(t) > 0|\alpha\}}{\alpha t/h}\right)^2 \cdot [B(c, \alpha, z)\,\mathrm{Var}\,\{A(t)|\alpha\} - \mathrm{Cov}\,\{A(t), O(t)|\alpha\}].$$

The approximation is complete when Γ is specified.

Numerical experimentation has shown that $\bar{B}$ is not very sensitive to the shape of $\Gamma(\alpha|\bar{\alpha}, v_d)$ for fixed values of $\bar{\alpha}$ and v_d. Accordingly, following Refs. 1 and 2, we have assumed that $\Gamma(\alpha|\bar{\alpha}, v_d)$ is a gamma distribution with mean $\bar{\alpha}$ and variance v_d, where $v_d = \mathrm{Var}\,(\alpha)$ is given by eq. (4). With these assumptions, the integrals in (9) can be computed numerically using a 51-point compound Simpson's rule. The accuracy of the approximation is discussed in Section III. In Section 2.3, we compare this approximation with the existing procedure.

2.3 The existing procedure

The existing approximation for average blocking is[1,2]

$$\bar{B}_e(c, \bar{\alpha}, z) \approx \int_0^\infty B(c, \alpha, z)d\Gamma(\alpha|\bar{\alpha}, v), \quad (10)$$

where

$$v = \mathrm{Var}\,(\hat{\alpha}) = 0.13\bar{\alpha}^{\phi}.$$

Consequently, there are two essential differences between the present method (10) and our approximation (9).

First, the existing method neglects the random component of load variation and assumes that all of the variation in the observed loads is due to day-to-day changes in the source load; i.e., the assumed load variation is too large by an amount $2\bar{\alpha}z/(t/h)$. The integral in (10) is an increasing function of v in the range of engineering interest (less than 5 percent average blocking) and, hence, $\bar{B}_e$ is larger than it would be if v_d were used in place of v.

Comparing (9) and (10), we see the second difference. The existing procedure implicitly uses $E\{[O(t)/A(t)]|\alpha\} \approx B(c, \alpha, z)$ and neglects the term $R(c, \alpha, z)$. Numerical experimentation has shown that $R(c, \alpha, z)$ is negative in the range of engineering interest and that $|R(c, \alpha, z)|$ is an increasing function of z. For z near 1, $R(c, \alpha, z)$ is negligible. However, for $z \geqq 2$ $R(c, \alpha, z)$ becomes significant. Consequently, $B(c, \alpha, z)$ is larger than $E\{[O(t)/A(t)]|\alpha\}$, and the difference increases with z.

The two differences combine to cause $\bar{B} < \bar{B}_e$ in all regions of engineering interest; i.e., the average of the observed single-hour blocking probabilities is less than that predicted by the existing method (10). Quantitative comparisons are made in Section III.

III. NUMERICAL RESULTS

To determine the accuracy of our approximation (6) relative to the existing procedure (10), a computer simulation was constructed for a full-access trunk group satisfying the assumptions specified in Section 2.2. The simulation was run for a large range of the system parameters c, $\bar{\alpha}$, z, and ϕ covering the regions of engineering interest ($B \approx 0.01$, $c \geqq 2$, $1 \leqq z \leqq 7$). Generally, a mean holding time of 180 seconds was used (although both smaller and larger values were used for sensitivity tests)* and 20-day averages were generated. All statistics are based on a sample size of 50; i.e., 1000 simulated hours.

3.1 Simulation output

Typical results from the simulation are summarized in Tables I and II. First consider Table I. The first four columns are the input parameters for the simulation. In order, they are the peakedness z, the offered load $\bar{\alpha}$, the trunk-group size c, and the conventional exponent ϕ defining the level of day-to-day variation. [That is, Var (α) was adjusted so as to produce the desired ϕ, as discussed in Section 2.2.] The next three columns give the simulated 20-day average blocking $\bar{B}_o$, the variance v_α of the daily offered source loads, and the (total) variance $v_{\hat{\alpha}}$ of the observed loads. The measurement variance $v_{\tilde{\alpha}}$ is then computed as $v_{\tilde{\alpha}} = v_{\hat{\alpha}} - v_\alpha$, and is compared with $2\bar{\alpha}z/(t/h)$ in the last two columns. In the cases shown, $2\bar{\alpha}z/(t/h)$ is an adequate approximation of the variance introduced by a finite measurement interval. The other

* The holding time was varied from 100 to 360 seconds and a small effect was observed. The effect is negligible for most engineering applications. Based on a survey of observed holding times, AT&T has requested that $h = 225$ seconds be used for the new traffic tables.

Table I — Simulation results

Simulation Input				Observed Data				
Peakedness (z)	Mean Load ($\bar{\alpha}$)	Trunks (c)	Variation Parameter (ϕ)	Average Blocking ($\bar{B}_o$)	Input Variance (v_α)	Observed Variance ($v_{\hat{\alpha}}$)	Residual Variance ($v_{\bar{\alpha}}$)	Theoretical Variance $\left(\frac{2\bar{\alpha}z}{t/h}\right)$
1.0	4.01	10	1.5	0.0083	0.72	1.03	0.31	0.40
4.0	17.80	40	1.5	0.0084	4.81	12.54	7.73	7.12
4.0	9.80	30	1.7	0.0049	3.60	6.61	3.01	3.92
7.0	9.75	40	1.84	0.0046	1.76	8.96	7.10	6.82

test cases indicated that the accuracy of the approximation generally increases as $\bar{\alpha}$ increases (as one would expect from the asymptotic nature of the approximation).

3.2 Existing approximation

The data in the first seven columns of Table II illustrate the size of the bias in the existing method. The first five columns of the table are the same as those in Table I. The next column $\bar{B}_e$ represents the computed average blocking corresponding to the source load, peakedness, trunk-group size, and respective day-to-day variance used in the simulation. The next column $\hat{c}_e$ gives the corresponding estimate (using the existing method) of the trunk-group size necessary to achieve the simulated blocking for the given input load, peakedness, and day-to-day variation parameter ϕ.

First, note the difference between $\bar{B}_0$ and $\bar{B}_e$. The existing approximation $\bar{B}_e$ is always larger than the actual average blocking $\bar{B}_o$, and the relative difference increases as z increases. The bias in $\bar{B}_e$ will cause the engineering estimates of trunks required (to meet objective

Table II — Engineering methods

Simulation					Engineering Methods			
Input				Output	Existing Method		New Method	
Peakedness (z)	Mean Load ($\bar{\alpha}$)	Trunks (c)	Variation Parameter (ϕ)	Simulated Average Blocking ($\bar{B}_o$)	Blocking Estimate ($\bar{B}_e$)	Trunk Estimate ($\hat{c}_e$)	Blocking Estimate ($\bar{B}$)	Trunk Estimate ($\hat{c}$)
1.0	4.01	10	1.5	0.0083	0.0100	10.24	0.0078	9.91
4.0	17.80	40	1.5	0.0084	0.0145	42.83	0.0083	39.97
4.0	9.80	30	1.7	0.0049	0.0103	32.96	0.0054	30.31
7.0	9.75	40	1.84	0.0046	0.0106	44.71	0.0045	39.93

service) to be too large. The bias in the trunk estimates is illustrated by comparing c with $\hat{c}_e$.*

In all the cases we considered, the bias was primarily a function of z. For $z = 1$, the bias was generally less than one trunk. For $z = 2$, $\hat{c}_e - c$ was usually between one and two trunks, and, for $z = 4$, the bias ranged between three and five trunks. The corresponding relative errors $(c - \hat{c}_e)/c$ were largest at the smaller values of c.

3.3 New approximation

The relative accuracy of the new approximation is illustrated in the last two columns of Table II. In these cases, the new approximation for $\bar{B}$ is much closer to the simulated blocking $\bar{B}_o$. The relative differences are quite small, especially when compared with the corresponding errors in the existing approximation. Since the estimate of $\bar{B}$ is good, the corresponding estimate $\hat{c}$ of trunks required to achieve $\bar{B}_o$ is quite close to c, the number actually required.

Similar results were obtained in all the other test cases. Accordingly, we conclude that the existing approximation (10) is biased, but the bias is essentially removed by using the new approximation (9).

Based on these results, the new approximation has been used to generate new trunk-engineering tables and algorithms for use by the Bell System.

IV. SUMMARY AND CONCLUSIONS

By combining a new model for day-to-day load variation with a new estimate of mean single-hour blocking, a new approximation was obtained for estimating the grade of service for probability-engineered trunk groups. Using this result, it has been possible to improve the accuracy of the presently recommended trunk-engineering procedures and thereby realize an increase in predicted trunk-group capacities. The increases are smaller for trunk groups serving Poisson traffic, but become substantial as the peakedness and level of day-to-day variation increase. The new approximation has been used to develop new trunk-engineering tables and algorithms that will soon be introduced into the Bell System.

APPENDIX A

Day-to-Day Load Variation

Let S denote a $GI/M/\infty$ service system which is observed over k disjoint intervals of time $\{I_1, \cdots, I_k\}$, each of length t.† The inter-

* Refer to Section 2.3 for a discussion of the individual components of the difference.

† The infinite server system is a standard model for traffic engineering. It is used to characterize a traffic process in terms of only interarrival times and service times.

arrival times during I_j are independent and identically distributed (iid) according to the probability distribution function F_j with mean

$$\frac{1}{\lambda_j} = \int_0^\infty t dF_j(t).$$

The arrival rates $\{\lambda_1, \cdots, \lambda_k\}$ are iid with mean $E(\lambda)$ and variance Var (λ). The peakedness (variance-to-mean ratio) of the arrival process is constant for all intervals. The service times are iid according to a negative-exponential distribution with mean h. The system is assumed to be operating in statistical equilibrium at the beginning of each interval; i.e., the initial point of each interval is a stationary point for the arrival process (see Ref. 3).

Let N_j be the number of arrivals during I_j and let ξ_{ij} denote the service time of the ith arrival in I_j. The average usage during $I = \sum_{j=1}^{k} I_j$ is estimated by*

$$\hat{u}_k = \frac{1}{k} \sum_{j=1}^{k} \sum_{i=1}^{N_j} \xi_{ij}. \tag{11}$$

(We assume the edge effects are negligible since S is in equilibrium during each interval I_j.) The corresponding estimate of average offered load is

$$\hat{a}_k = \frac{1}{t} \hat{u}_k. \tag{12}$$

In this section, we obtain the mean and variance of $\hat{a}_k$.

From Ref. 5 and eqs. (11) and (12), we have

$$\begin{aligned} E(\hat{a}_k) &= \frac{1}{kt} \sum_{j=1}^{k} \underset{N_j}{E} \left\{ E \left\{ \sum_{i=1}^{N_j} \xi_{ij} | N_j \right\} \right\}, \\ &= \frac{1}{kt} \sum_{j=1}^{k} \underset{N_j}{E} \{hN_j\} \\ &= \frac{h}{kt} \sum_{j=1}^{k} E\{N_j\}. \end{aligned} \tag{13}$$

Since the beginning of each interval is a stationary point for the arrival process, it follows that $E\{N_j | \lambda_j\} = \lambda_j t$ (see Ref. 6). Thus, for the arrival process, it follows that

$$\begin{aligned} E\{N_j\} &= \underset{\lambda_j}{E} \{E\{N_j | \lambda_j\}\} \\ &= \underset{\lambda_j}{E} \{\lambda_j t\} \\ &= E\{\lambda\} t \end{aligned}$$

* Our studies have shown that the additional measurement variance caused by discretely sampling the usage with a 100-second-scan Traffic Usage Recorder was negligible when compared with the variance caused by the load variation.

and so from (13) we have

$$E\{\hat{\alpha}_k\} = E\{\lambda\}h.$$

Since the ξ_{ij} are independent, eqs. (11) and (12) yield

$$\operatorname{Var}(\hat{\alpha}_k) = \frac{1}{(kt)^2}\sum_{j=1}^{k} \operatorname{Var}\left\{\sum_{i=1}^{N_j} \xi_{ij}\right\}, \tag{14}$$

and from Ref. 5,

$$\operatorname{Var}\left\{\sum_{i=1}^{N_j} \xi_{ij}\right\} = \underset{\lambda_j}{E}\left\{\underset{N_j}{\operatorname{Var}}\left\{\sum_{i=1}^{N_j} \xi_{ij}\middle|\lambda_j\right\}\right\} + \underset{\lambda_j}{\operatorname{Var}}\left\{\underset{N_j}{E}\left\{\sum_{i=1}^{N_j} \xi_{ij}\middle|\lambda_j\right\}\right\}. \tag{15}$$

We first expand

$$\begin{aligned}
\underset{N_j}{\operatorname{Var}}\left\{\sum_{i=1}^{N_j} \xi_{ij}\middle|\lambda_j\right\} &= \underset{N_j}{\operatorname{Var}}\left\{E\left\{\sum_{i=1}^{N_j} \xi_{ij}\middle|N_j, \lambda_j\right\}\right\} \\
&\qquad + \underset{N_j}{E}\left\{\operatorname{Var}\left\{\sum_{i=1}^{N_j} \xi_{ij}\middle|N_j, \lambda_j\right\}\right\} \\
&= \underset{N_j}{\operatorname{Var}}\{N_j h|\lambda_j\} + \underset{N_j}{E}\{N_j h^2|\lambda_j\} \\
&= h^2\lambda_j t\left(1 + \frac{\operatorname{Var}\{N_j|\lambda_j\}}{E\{N_j|\lambda_j\}}\right).
\end{aligned} \tag{16}$$

Let z denote the traffic peakedness (which is constant over I). Then from Ref. 7, we have the approximation

$$\frac{\operatorname{Var}\{N_j|\lambda_j\}}{E\{N_j|\lambda_j\}} \approx 2z - 1; \tag{17}$$

the approximation has been found to be quite good for $\alpha > (z-1)$ and $t \geqq 10h$ (and probably acceptable for engineering purposes for $t \geqq 5h$). Substituting (17) into (16), we obtain

$$\underset{N_j}{\operatorname{Var}}\left\{\sum_{i=1}^{N_j} \xi_{ij}\middle|\lambda_j\right\} = 2zh^2 t\lambda_j. \tag{18}$$

Expanding the second term in (15) in the same manner as in eq. (13) yields

$$\underset{N_j}{E}\left\{\sum_{i=1}^{N_j} \xi_{ij}\middle|\lambda_j\right\} = ht\lambda_j. \tag{19}$$

Substituting (18) and (19) in (15) provides

$$\operatorname{Var}\left\{\sum_{i=1}^{N_j} \xi_{ij}\right\} = 2zh^2 tE\{\lambda\} + (ht)^2 \operatorname{Var}\{\lambda\}, \tag{20}$$

which, with eq. (14), implies that

$$\operatorname{Var}(\hat{\alpha}_k) = \frac{1}{k}\left(\frac{2zhE\{\lambda\}}{t/h} + h^2 \operatorname{Var}\{\lambda\}\right). \tag{21}$$

The offered load during I_j is $\alpha_j = \lambda_j h$. Hence, $E\{\alpha_j\} = hE\{\lambda\}$ and $\text{Var}\{\alpha_j\} = h^2 \text{Var}\{\lambda\}$. Dropping the subscript j, we have

$$\text{Var}\{\hat{\alpha}_k\} = \frac{1}{k}\left(\frac{2zE\{\alpha\}}{t/h} + \text{Var}\{\alpha\}\right). \tag{22}$$

Equation (22) has a simple interpretation. The first term $(2zE\{\alpha\})/(t/h)$ represents the component (of the observed variation) which is due to a finite measurement period. The second component $\text{Var}\{\alpha\}$ is the "true" (day-to-day) variation of the offered source load. The factor k results from averaging k observations. Setting $k = 1$ and $\bar{\alpha} = E\{\alpha\}$, we have the variance of the single-hour load estimate

$$\text{Var}\{\hat{\alpha}\} = \frac{2z\bar{\alpha}}{t/h} + \text{Var}\{\alpha\}, \tag{23}$$

where the individual terms have the interpretations noted above.

APPENDIX B

Probability of an Arrival

In this appendix, we derive an expression for $\Pr\{A(t) > 0 | \alpha\}$. We assume that the interarrival times are independent and identically distributed according to the distribution function F. We further assume that F is a mixture of exponentials, with the parameters chosen so that F approximates the interarrival distribution of an overflow process with load α and peakedness z (see Ref. 8); that is,

$$F(t) = 1 - k_1 e^{-r_1 t} - k_2 e^{-r_2 t}.$$

Now, $\Pr\{A(t) > 0 | \alpha\}$ is just the probability that the first arrival after a random entry point (i.e., a stationary point for the arrival process) will occur before t units of time have elapsed. Thus, from Ref. 9, we have

$$\begin{aligned}\Pr\{A(t) > 0 | \alpha\} &= \alpha \int_0^t [1 - F(x)]dx \\ &= \alpha\left(\frac{k_1}{r_1} + \frac{k_2}{r_2} - \frac{k_1}{r_1} e^{-r_1 t} - \frac{k_2}{r_2} e^{-r_2 t}\right).\end{aligned}$$

Since $\int_0^\infty [1 - F(x)]dx = 1/\alpha$, it follows that

$$\Pr\{A(t) > 0 | \alpha\} = 1 - \alpha\left(\frac{k_1}{r_1} e^{-r_1 t} + \frac{k_2}{r_2} e^{-r_2 t}\right),$$

where the parameters k_i and r_i are functions of α and z as specified in Ref. 8.

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